Multivariable Newton-Puiseux Theorem for Convergent Generalised Power Series

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Abstract

A generalised power series (in several variables) is a series with real nonnegative exponents whose support is contained in a cartesian product of well-ordered subsets of $\mathbb{R}^{\geq 0}$ (see [DS98]). Let $\mathcal{A}$ be the collection of all convergent generalised power series. We show that, if $f(x_1, \ldots, x_m, y) \in \mathcal{A}$, then the solutions $y = \varphi(x_1, \ldots, x_m)$ of the equation $f = 0$ belong to the compositional closure of $\mathcal{A} \cup \{x \mapsto 1/x\}$. The same result holds if $\mathcal{A}$ is the collection $\left\{\mathcal{Q}_{m,n,r} : \ m, n \in \mathbb{N}, \ r \in (0, \infty)^{m+n}\right\}$ in [KRS09, Section 7], which contains the Dulac Transition Maps of real analytic planar vector fields in a neighbourhood of hyperbolic non-resonant singular points. A similar result holds if $\mathcal{A}$ is a quasianalytic Denjoy-Carleman class or if $\mathcal{A}$ is the collection $\mathcal{G}$ in [DS00, Section 2], which is a class of Gevrey functions in several variables.

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The Newton-Puiseux Theorem states that, if $f(x, y)$ is an analytic germ in two variables, then the solutions $y = \varphi(x)$ of the equation $f = 0$ can be expanded as Puiseux series that are convergent in a neighbourhood of the origin (see for example [BK86]). A multivariable version of this result in the real case states that, if $f(x_1, \ldots, x_m, y)$ is a real analytic germ, then, after a finite sequence of blow-ups with centre a real analytic manifold, the solutions $y = \varphi(x_1, \ldots, x_m)$ of the equation $f = 0$ are analytic in a neighbourhood of the origin (see for example [Par01, Theorem 4.1]). An equivalent formulation states that the solutions $y = \varphi(x_1, \ldots, x_m)$ in a neighbourhood of the origin are obtained, piecewise, as finite compositions of analytic functions, taking n^{th} roots and quotients (see for example [DMM94, Corollary 2.15] and [LR97, Theorem 1]).

Here we extend this result to four different settings, which all generalise in some way the real analytic framework. In all the settings under consideration, we already have a local uniformisation result [RSW03, MVSSR12, RS13] which allows to parametrise the zero set of a function in the class. Our aim here is to refine this procedure, in the spirit of the elimination result in [DD88], in the following way: given a function $f(x, y)$ in the class under consideration, we provide a uniformisation algorithm which "respects" the variable y and hence allows to solve the equation $f = 0$ with respect to y.

The classes of functions we consider are the following.

a) Let $M = (M_0, M_1, \ldots)$ be an increasing sequence of positive real numbers (with $M_0 \geq 1$) and $B \subseteq \mathbb{R}^m$ be a compact box. We assume that M is strongly log-convex and we consider the Denjoy-Carleman algebra of functions $\mathcal{C}_B(M)$ defined in [RSW03]. This is an algebra of functions $f : B \to \mathbb{R}$ which each extend to a $\mathcal{C}^\infty$ function on some open neighbourhood $U \supseteq B$ and whose derivatives satisfy a certain type of bounds depending on M (see [RSW03, p. 751]). $\mathcal{C}_B(M)$ is *quasianalytic* if $\sum_{i \in \mathbb{N}} \frac{M_i}{M_{i+1}} = \infty$. The functions in $\mathcal{C}_B(M)$ are not analytic in general, however, if $\mathcal{C}_B(M)$ is quasianalytic, then for every $x \in B$, the algebra morphism which associates to $f \in \mathcal{C}_B(M)$ its (divergent) Taylor expansion at x is injective. The quasianalytic Denjoy-Carleman class $\mathcal{C}(M)$ is the collection $\{\mathcal{C}_B(M) : \ m \in \mathbb{N}, \ B \subseteq \mathbb{R}^m \text{ compact box}\}$.

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b) A (formal) generalised power series in m variables $X = (X_1, \ldots, X_m)$ is a series $F(X) = \sum_\alpha c_\alpha X^\alpha$ such that $\alpha \in [0,\infty)^m$, $c_\alpha \in \mathbb{R}$ and there are well-ordered subsets $S_1, \ldots, S_m \subseteq [0,\infty)$ such that the support of F is contained in $S_1 \times \ldots \times S_m$ (see [DS98]). The series F is convergent if there is a polyradius $r = (r_1, \ldots, r_m) \in (0,\infty)^m$ such that $\sum_\alpha |c_\alpha| r^\alpha < \infty$. A convergent generalised power series gives rise to a real-valued function $F(x) = \sum c_\alpha x^\alpha \in \mathbb{R}\{x^*\}_r$, which is continuous on $[0,r_1) \times \ldots \times [0,r_m)$ and analytic on the interior of its domain. We denote by $\mathbb{R}[\![X^*]\!]$ the algebra of all formal generalised power series and consider the algebra $\mathbb{R}\{x^*\} = \bigcup_{r\in(0,\infty)^m} \mathbb{R}\{x^*\}_r$ of all convergent generalised power series. Examples of convergent generalised power series are the function $\zeta(-\log x) = \sum_{n=1}^\infty x^{\log n}$ (where ζ is the Riemann zeta function) and the solution $f(x) = \sum_{n,i=0}^\infty \frac{1}{2^i} x^{2+n-\frac{1}{2^i}}$ of the functional equation $(1-x) f(x) = x + \frac{1}{2}x(1-\sqrt{x}) f(\sqrt{x})$.

c) For $R = (R_1, \ldots, R_m) \in (0,\infty)^m$ a polyradius, we consider the algebra $\mathcal{G}(R)$ defined in [DS00, Definition 2.20]. Its elements are $\mathcal{C}^\infty$ functions defined on $[0,R_1] \times \ldots \times [0,R_m]$ which satisfy a Gevrey condition. By a known result in multisummability theory, these algebras satisfy the following quasianalyticity condition: the morphism, which associates the germ at zero of a function in $\mathcal{G}(R)$ its (divergent) Taylor expansion at the origin, is injective (see [DS00, Proposition 2.18]). We let $\mathcal{G}$ be the collection $\{\mathcal{G}(R) : \ m \in \mathbb{N}, \ R \in (0,\infty)^m\}$. This collection contains for example the function $\psi(x)$ appearing in Binet's second formula, i.e. such that $\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log(x) + \frac{1}{2}\log(2\pi) + \psi\left(\frac{1}{x}\right)$, where Γ is Euler's Gamma function (see for example [DS00, Example 8.1]).

d) For $r \in (0,\infty)^{m+n}$ a polyradius, we consider the algebra $\mathcal{Q}_{m,n,r}$ defined in [KRS09, Definition 7.1]. Its elements are continuous real-valued functions which have a holomorphic extension to some "quadratic domain" $U \subseteq \mathbf{L}^{m+n}$, where $\mathbf{L}$ is the Riemann surface of the logarithm. One can define a morphism T which associates to the germ f of a function in $\mathcal{Q}_{m,n,r}$ an *asymptotic expansion* $T(f) \in \mathbb{R}[\![X^*]\!]$. It is shown in [KRS09, Proposition 2.8], using results of Ilyashenko's in [Il'91], that the morphism T is injective (quasianalyticity). We let $\mathcal{Q}$ be the collection $\left\{\mathcal{Q}_{m,n,r} : \ m,n \in \mathbb{N}, \ r \in (0,\infty)^{m+n}\right\}$. The motivation for looking at this type of algebras is that they contain the Dulac transition maps of real analytic planar vector fields in a neighbourhood of hyperbolic non-resonant singular points.

Let $\mathcal{A}$ be a collection of real-valued functions. An $\mathcal{A}$-*term* is defined inductively as follows. An $\mathcal{A}$-term of depth zero is an element of $\mathcal{A}$. Let $x = (x_1, \ldots, x_m)$. A function $f(x)$ is an $\mathcal{A}$-term of depth $\leq k$ if there exist $m \in \mathbb{N}$, $g \in \mathcal{A}$ and $\mathcal{A}$-terms $t_1(x), \ldots, t_m(x)$ of depth $\leq k-1$ such that $f(x) = g(t_1(x), \ldots, t_m(x))$.

Let $r \in (0,\infty)^m$ be a polyradius and let I_r be either the set $(-r_1, r_1) \times \ldots \times (-r_m, r_m)$ or the set $[0, r_1) \times \ldots \times [0, r_m)$.

A set $C \subseteq \mathbb{R}^m$ is an $\mathcal{A}$-*base* if there are a polyradius $r \in \mathbb{R}^m$ and $\mathcal{A}$-terms $t_0, t_1, \ldots, t_q$ defined on I_r, such that

$$C = \{x \in I_r : \ t_0(x) = 0, \ t_1(x) > 0, \ \ldots, t_q(x) > 0\}.$$

A set $D \subseteq \mathbb{R}^{m+1}$ is an $\mathcal{A}$-*cell* if there are an $\mathcal{A}$-base $C \subseteq \mathbb{R}^m$ and terms $t_1(x), t_2(x)$ in m variables such that

$$\begin{aligned} &\text{either } D = \{(x,y) : \ x \in C, \ y = t_1(x)\}, \\ &\text{or } D = \{(x,y) : \ x \in C, \ t_1(x) < y < t_2(x)\}. \end{aligned}$$

If $A \subseteq W \subseteq \mathbb{R}^{m+1}$, then an $\mathcal{A}$-*cell decomposition of* W *compatible with* A is a finite partition of W into $\mathcal{A}$-cells such that every $\mathcal{A}$-cell in the partition is either contained in A or disjoint from A.

The collections $\mathcal{A}$ that we consider in this paper are collections of functions definable in some o-minimal expansion of the real field (see [Dri98] for the definition and basic properties of o-minimal structures). In particular, there is a well-defined notion of dimension for $\mathcal{A}$-bases and $\mathcal{A}$-cells (which coincides with the topological notion). Though not necessarily connected, an $\mathcal{A}$-cell has, by o-minimality, a finite number of connected components.

We can now state our main results.

Theorem A. *Let $\mathcal{C}(M)$ be a quasianalytic Denjoy-Carleman class and let $\mathcal{A} = \mathcal{C}(M) \cup \left\{x \mapsto \frac{1}{x}\right\} \cup \{x \mapsto \sqrt[n]{x} : n \in \mathbb{N}\}$. Let $x = (x_1, \ldots, x_m)$ and $f(x, y) \in \mathcal{C}_B(M)$ for some compact box $B \subseteq \mathbb{R}^{m+1}$ with $0 \in \overset{\circ}{B}$. Then there exist a neighbourhood $W \subseteq B$ of the origin and an $\mathcal{A}$-cell decomposition of W compatible with the set $\{(x, y) \in W : f(x, y) = 0\}$.*

We will show that Theorem A also holds if we replace $\mathcal{C}(M)$ by any collection $\mathcal{C}$ of quasianalytic algebras containing all polynomials and closed under composition, implicit function and monomial division (see [RSW03, Section 3]).

Theorem B. *Let $\mathcal{A} = \left\{\mathbb{R}\left\{(x_1, \ldots, x_m)^*\right\} : m \in \mathbb{N}\right\} \cup \left\{x \mapsto \frac{1}{x}\right\}$. Let $x = (x_1, \ldots, x_m)$ and $f(x, y) \in \mathbb{R}\left\{(x, y)^*\right\}_r$ for some polyradius $r \in (0, \infty)^{m+1}$. Then there exist a neighbourhood $W \subseteq \mathbb{R}^{m+1}$ of the origin and an $\mathcal{A}$-cell decomposition of $W \cap I_r$, where $I_r = [0, r_1) \times \ldots \times [0, r_{m+1})$, which is compatible with the set $\{(x, y) \in W \cap I_r : f(x, y) = 0\}$.*

Finally, we show that the proof of Theorem B can be easily adapted to prove the following two results.

Theorem C. *Let $\mathcal{A} = \mathcal{G} \cup \left\{x \mapsto \frac{1}{x}\right\} \cup \{x \mapsto \sqrt[n]{x} : n \in \mathbb{N}\}$, where $\mathcal{G}$ is as in [DS00, Section 2]. Let $x = (x_1, \ldots, x_m)$ and $f(x, y) \in \mathcal{G}(R)$ for some polyradius $R \in (0, \infty)^{m+1}$. Then there exist a neighbourhood $W \subseteq \mathbb{R}^{m+1}$ of the origin and an $\mathcal{A}$-cell decomposition of $W \cap I_R$, where $I_R = [0, R_1) \times \ldots \times [0, R_{m+1})$, which is compatible with the set $\{(x, y) \in W \cap I_R : f(x, y) = 0\}$.*

Theorem D. *Let $\mathcal{A} = \{\mathcal{Q}_{m,0,r} : m \in \mathbb{N}, r \in (0, \infty)^m\} \cup \left\{x \mapsto \frac{1}{x}\right\}$, where $\mathcal{Q}_{m,0,r}$ is as in [KRS09, Definition 7.1]. Let $x = (x_1, \ldots, x_m)$ and $f(x, y) \in \mathcal{Q}_{m+1,0,r}$ for some polyradius $r \in (0, \infty)^{m+1}$. Then there exist a neighbourhood $W \subseteq \mathbb{R}^{m+1}$ of the origin and an $\mathcal{A}$-cell decomposition of $W \cap I_r$, where $I_r = [0, r_1) \times \ldots \times [0, r_{m+1})$, which is compatible with the set $\{(x, y) \in W \cap I_r : f(x, y) = 0\}$.*

Theorems A,B,C and D immediately imply that in each case the solutions of the equation $f(x, y) = 0$ have the form $\varphi : C \to \mathbb{R}$, where $C \subseteq \mathbb{R}^m$ is an $\mathcal{A}$-base and $\varphi(x)$ is an $\mathcal{A}$-term. The proofs follow the same strategy, which we now illustrate briefly. In analogy with the real analytic case, we define a class of blow-up transformations adapted to the functions under consideration. We show that, after a finite sequence of such transformations, the germ at zero of f is normal crossing. The transformations we use *respect* the variable y in the following way: if $\rho : \mathbb{R}^{m+1} \ni (x', y') \mapsto (x, y) \in \mathbb{R}^{m+1}$ is one of such transformations and we know how to solve explicitly the equation $f \circ \rho(x', y') = 0$ with respect to the variable y', then we also know how to solve explicitly the equation $f(x, y) = 0$ with respect to the variable y. Moreover, such transformations are bijective outside a set of small dimension and the components of the inverse map, when defined, are $\mathcal{A}$-terms.

The desingularisation procedure which allows to reduce to the case when f is normal crossing exploits a fundamental common property of the algebras of functions under consideration: quasi-analyticity. This property is tautological in case b) and follows by classical theorems in cases a), c) and d) (see [Rud87, Tou94, Il'91]). It allows to deduce the wanted result for f from a formal monomialisation algorithm for the series $\hat{f}(X, Y)$, where $\hat{f}$ is the Taylor expansion at zero of f in cases a) and c), the generalised power series $f(X, Y)$ in case b) and the asymptotic expansion $T(f)$ of the germ at zero of f in case d). The monomialisation algorithm we exhibit differs from the ones in [RSW03, MVSSR12] in that it respects the variable Y and hence allows to solve the equation $f = 0$.

Theorems A,B,C and D could also be deduced from a general quantifier elimination result in [RS13]. However, the solving process described in [RS13] is not algorithmic, since it uses a highly nonconstructive result, namely an o-minimal Preparation Theorem in [DS02]. Here instead we deduce the explicit form of the solutions of $f = 0$ solely from the analysis of the Newton polyhedron of $\hat{f}$.

1 Proof of Theorem A

For every $B = [a_1, b_1] \times \ldots \times [a_m, b_m]$ a compact box in $\mathbb{R}^m$, let $\mathcal{C}_B$ be an $\mathbb{R}$-algebra of functions $f : B \to \mathbb{R}$ such that f extends to a $\mathcal{C}^\infty$ function on some open set $U \supseteq B$. Suppose that the collection $\mathcal{C} = \{\mathcal{C}_B :\ m \in \mathbb{N},\ B \subseteq \mathbb{R}^m \text{ compact box}\}$ contains all the polynomials with real coefficients and is closed under composition, taking implicit functions and monomial division. More precisely, suppose that $\mathcal{C}$ satisfies conditions (C1)-(C3) in [RSW03, Section 3]. Let $\mathcal{C}_{0,m}$ be the algebra of all the germs at the origin of the functions in $\mathcal{C}_m := \left\{\mathcal{C}_B :\ B \subseteq \mathbb{R}^m,\ 0 \in \mathring{B}\right\}$. We require that $\mathcal{C}_0 := \{\mathcal{C}_{0,m} :\ m \in \mathbb{N}\}$ satisfy conditions (C6) and (C7) in [RSW03, Section 3] (condition (C4) follows automatically).

Finally, we suppose that $\mathcal{C}_0$ is quasianalytic (condition (C5)), i.e. the morphism $\hat{}: \mathcal{C}_{0,m} \to \mathbb{R}\llbracket X_1, \ldots, X_m \rrbracket$ which associates to a germ f its Taylor expansion $\hat{f}$ at the origin is *injective*. We denote by $\hat{\mathcal{C}}\langle X_1, \ldots, X_m\rangle \subseteq \mathbb{R}\llbracket X_1, \ldots, X_m \rrbracket$ the image of $\mathcal{C}_{0,m}$ under the morphism $\hat{}$.

It is shown in [RSW03] that, if $\mathcal{C}(M)$ is a quasianalytic Denjoy-Carleman class, where M is strongly log-convex, then $\mathcal{C}(M)$ satisfies all the above conditions. Another example of collection of functions satisfying the above conditions is given by the collection of all $\mathcal{A}_H$-analytic functions as in [RSS07], where $H = (H_1, \ldots, H_r) : (-\varepsilon, \varepsilon) \to \mathbb{R}^r$ is a solution of a first order analytic differential equation which is singular at the origin.

The first step is to give a monomialisation algorithm for elements of $\hat{\mathcal{C}}\langle X_1, \ldots, X_m\rangle$, which is a modification of the one in [RSW03, Theorem 2.5]. Our monomialisation tools will be the same admissible substitutions used in [RSW03]. However, keeping in mind our final aim, i.e. the proof of Theorem A and the more general method required for the proof of Theorem B, we will fix our own notation, as follows.

Let $X = (X_1, \ldots, X_m)$ and $X' = (X'_1, \ldots, X'_m)$. We consider a collection of formal transformations which map X' to X. Each transformation $X' \mapsto X$ will induce an $\mathbb{R}$-algebra homomorphism $\mathbb{R}\llbracket X \rrbracket \to \mathbb{R}\llbracket X' \rrbracket$ by composition.

Definition 1.1. An *elementary transformation* is a map $\nu : X' \to X$ of either of the following forms.

- A *ramification*: for $1 \le i \le m$ and $d \in \mathbb{N}$, let $r_i^{d,\pm}(X') = X$, where $X_j = X'_j$ for $j \neq i$ and $X_i = \pm X_i'^d$.
- A *Tschirnausen translation*: for $H \in \hat{\mathcal{C}}\left\langle X'_1, \ldots, X'_{m-1}\right\rangle$, with $H(0) = 0$, let $\tau_H(X') = X$, where $X_j = X'_j$ for $j \neq m$ and $X_m = X'_m + H\left(X'_1, \ldots, X'_{m-1}\right)$.
- A *linear transformation*: for $1 \le i \le m$ and $c = (c_1, \ldots, c_{i-1}) \in \mathbb{R}^{i-1}$, let $L_{i,c}(X') = X$ where $X_j = X'_j + c_j X'_i$ for $j < i$ and $X_j = X'_j$ for $j \ge i$.
- A *blow-up chart*: for $1 \le i, j \le m$ and $\lambda \in \mathbb{R}$, let $\pi_{i,j}^\lambda(X') = X$, where $X_k = X'_k$ for $k \neq i$ and $X_i = X'_j(\lambda + X'_i)$. Moreover, let $\pi_{i,j}^\infty = \pi_{j,i}^0$.

An *admissible transformation* $\rho : X' \mapsto X$ is a finite composition of elementary transformations. Note that, if $F \in \hat{\mathcal{C}}\langle X\rangle$, then $F \circ \rho \in \hat{\mathcal{C}}\langle X'\rangle$.

An *elementary family* is either of the following collections of elementary transformations: $\left\{r_i^{d,+}, r_i^{d,-}\right\}$, $\{\tau_H\}$, $\{L_{i,c}\}$ or $\left\{\pi_{i,j}^\lambda :\ \lambda \in \mathbb{R} \cup \{\infty\}\right\}$. An *admissible family* is defined inductively. An admissible family of length 1 is an elementary family. An admissible family $\mathcal{F}$ of length $\le q$ is obtained from an elementary family $\mathcal{F}_0$ in the following way: for all $\nu \in \mathcal{F}_0$, let $\mathcal{F}_\nu$ be an admissible family of length $\le q-1$ and define $\mathcal{F} = \{\nu \circ \rho' :\ \nu \in \mathcal{F}_0,\ \rho' \in \mathcal{F}_\nu\}$.

Finally, we say that a series $F \in \mathbb{R}\llbracket X \rrbracket$ has a certain property P *after admissible family* if there exists an admissible family $\mathcal{F}$ such that for every $\rho \in \mathcal{F}$ the series $F \circ \rho(X')$ has the property P.

1.2. A series $F \in \hat{\mathcal{C}}\langle X\rangle$ is *normal* if there are $\alpha \in \mathbb{N}^m$ and a unit $U \in \left(\hat{\mathcal{C}}\langle X\rangle\right)^\times$ such that $F(X) = X^\alpha U(X)$.

Let $X = (X_1, \dots, X_m)$ and $F \in \hat{\mathcal{C}}\langle X, Y\rangle$. We show that, after admissible family, F is normal.

Step 1: regularity. The first step is to show that, after admissible family, there are $\alpha \in \mathbb{N}^m$ and $G \in \hat{\mathcal{C}}\langle X, Y\rangle$ such that $F(X,Y) = X^\alpha G(X,Y)$ and G is regular of some order $d \in \mathbb{N}$ in the variable Y, i.e. there is a unit $U \in \hat{\mathcal{C}}\langle Y\rangle^\times$ such that $G(0,Y) = Y^d U(Y)$.

This could easily be achieved by performing a linear transformation involving the variables (X,Y). However, such a transformation would not respect the variable Y. Hence we follow a different strategy. Let $F(X,Y) = \sum_{i\in\mathbb{N}} F_i(X) Y^i$, where $F_i(X) = \frac{1}{i!}\frac{\partial^i F}{\partial Y^i}(X,0) \in \hat{\mathcal{C}}\langle X\rangle$. By Noetherianity of $\mathbb{R}[\![X]\!]$, the $\mathbb{R}[\![X]\!]$-ideal generated by the family $\mathcal{G} = \{F_i(X) : i \in \mathbb{N}\}$ is finitely generated. Let $\{F_0, \dots, F_q\}$ be a set of generators. We obtain the following (formal) finite presentation

$$F(X,Y) = \sum_{i=0}^{q} F_i(X) Y^i U_i(X,Y), \tag{*}$$

where the units $U_i \in \mathbb{R}[\![X,Y]\!]^\times$ do not necessarily belong to $\hat{\mathcal{C}}\langle X, Y\rangle$. However, we can apply the monomialisation algorithm in [RSW03, Theorem 2.5] simultaneously to the generators $F_0, \dots, F_q$ and obtain that, after admissible family, the generators are normal and linearly ordered by division. Let $d \in \{0, \dots, q\}$ be smallest such that F_d divides all the generators. Since F_d is in normal form, let $\alpha \in \mathbb{N}^m$ and $V \in \hat{\mathcal{C}}\langle X\rangle^\times$ be such that $F_d(X) = X^\alpha V(X)$. Then $F(X,Y) = X^\alpha G(X,Y)$, where the series G belongs to $\hat{\mathcal{C}}(X,Y)$, by monomial division, and G is regular of order d.

Remark 1.3. The above argument also shows that in the real analytic setting, in order to obtain regularity in a chosen variable Y, there is no need to prove a convergent version of the finite presentation in (*). In particular, in Denef and van den Dries' proof of quantifier elimination for the real field with restricted analytic functions and the function $x \mapsto 1/x$, one can dispense with the use of faithful flatness, as in [DD88, Lemma 4.12].

Step 2: decreasing the order of regularity. Suppose that $d > 1$. The next step is to show that, after admissible family, either the order of regularity of G has decreased or F is normal.

This can be achieved by following the algorithm in the proof of [RSW03, Theorem 2.5], which we briefly summarise. By the Taylor formula, there are series $A_1, \dots, A_d \in \hat{\mathcal{C}}\langle X\rangle$, with $A_i(0) = 0$, and a unit $U \in \hat{\mathcal{C}}\langle X, Y\rangle^\times$ such that

$$G(X,Y) = A_d(X) + \ldots + A_1(X) Y^{d-1} + U(X,Y) Y^d.$$

After a Tschirnhausen translation, we may assume that $A_1 = 0$. After admissible family, we may suppose that the A_i are normal, i.e. $A_i(X) = X^{\beta_i} U_i(X)$ for some $\beta_i \in \mathbb{N}^m$ and $U_i \in \hat{\mathcal{C}}\langle X\rangle^\times$. Moreover, the components of β_i are divisible by i and for some $l \in \{2, \dots, d\}$, the series $A_l^{1/l}$ divides all the series $A_i^{1/i}$. Let $j \in \{1, \dots m\}$ be such that the variable X_j appears with a nonzero exponent in the monomial X^{β_l} and consider the family of blow-up transformations $\left\{\pi^\lambda_{m+1,j} : \lambda \in \mathbb{R} \cup \{\infty\}\right\}$. After the transformation $\pi^\infty_{m+1,j}$, the series G has the form $Y^d V(X,Y)$, where $V \in \hat{\mathcal{C}}\langle X, Y\rangle^\times$, so in this case F is normal. After the transformation $\pi^0_{m+1,j}$, the exponent of X_j in the monomial X^{β_l} has decreased by the quantity l. For $\lambda \in \mathbb{R} \setminus \{0\}$, after the transformation $\pi^\lambda_{m+1,j}$, thanks to the fact that $A_1 = 0$, the order of G is at most $d-1$. Hence we obtain the required result after an admissible family of finite compositions of blow-up transformations.

Notice that, if G is regular of order 1 in the variable Y, then we can translate Y by the solution to the implicit function problem $G = 0$ and obtain that F is normal. This concludes the proof of 1.2.

1.4. We show that the above formal monomialisation process allows to solve explicitly an equation of the form $f(x, y) = 0$, where $f \in \mathcal{C}$.

Remark 1.5. If $r = (r_1, \ldots, r_m) \in (0, \infty)^m$ is a polyradius, we set $I_r = (-r_1, r_1) \times \ldots \times (-r_m, r_m)$. A (formal) admissible transformation $\rho : X' \mapsto X$ clearly induces a geometric transformation $\rho : x' \mapsto x$ of the space $\mathbb{R}^m$. More precisely, if $\mathcal{F}$ is an admissible family, then for all polyradius $r \in (0, \infty)^m$ there exist a polyradius $r' \in (0, \infty)^m$ and an open neighbourhood $W \subseteq \mathbb{R}^m$ of the origin such that for every $\rho \in \mathcal{F}$ we have $\rho(I_{r'}) \subseteq I_r$ and, by a standard compactness argument, there is a finite subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $W = \bigcup_{\rho \in \mathcal{F}_0} \rho(I_{r'})$.

Let $B \subseteq \mathbb{R}^{m+1}$ be a compact box with $0 \in \mathring{B}$ and $f(x, y) \in \mathcal{C}_B$, where $x = (x_1, \ldots, x_m)$. It follows from the above remark and the quasianalyticity property that there are an open neighbourhood $W \subseteq \mathbb{R}^{m+1}$ of the origin and an admissible family $\mathcal{F}$ of admissible geometric transformations $\rho : I_{r'} \to I_r$ (where $r, r' \in (0, \infty)^{m+1}$), such that for some finite subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ we have $W = \bigcup_{\rho \in \mathcal{F}_0} \rho(I_{r'})$ and for every $\rho \in \mathcal{F}_0$, the function $f \circ \rho$ is normal, i.e. there are $\alpha \in \mathbb{N}^{m+1}$ and a unit $U(x', y') \in \left(\mathcal{C}_{\bar{I}_{r'}}\right)^{\times}$ such that $f \circ \rho(x', y') = (x'y')^{\alpha} U(x', y')$. The sets $\rho(I_{r'})$ are $\mathcal{A}$-cells.

Notice that all the elementary transformations $\nu : I_{r'} \to I_r$ used in the proof of 1.2, except for the singular blow-up chart $\pi^{\infty}_{m+1,j}$, respect the variable y is the following sense: the map $\hat{\nu} = (\nu_1, \ldots, \nu_m)$ does not depend on y' and is an elementary transformation of $\mathbb{R}^m$, hence we can write $x = \hat{\nu}(x')$. Moreover, there exist sets $S, S' \subseteq \mathbb{R}^m$ (which are either empty or of the form $\{x_i = 0\}$ for some $i \in \{1, \ldots, m\}$) such that the map $\hat{\nu} \restriction (-r'_1, r'_1) \times \ldots \times (-r'_m, r'_m) \setminus S$ is a bijection onto its image and for all $x' \in (-r'_1, r'_1) \times \ldots \times (-r'_m, r'_m) \setminus S'$ the map $y' \mapsto \nu_{m+1}(x', y')$ is a bijection onto its image. The components of the inverse maps $x \mapsto x'$ and $(x', y) \mapsto y'$ are $\mathcal{C}$-terms.

We can now conclude the proof of Theorem A by induction on the pairs (l, m), where l is the length of an admissible family $\mathcal{F}$ which monomialises f. Notice that if f is normal then the $\mathcal{A}$-cell decomposition required is trivial. Suppose that $\mathcal{F}$ is obtained from some elementary family $\mathcal{F}_0$ by composing every $\nu : I_{r'} \to I_r \in \mathcal{F}_0$ with all the elements of a suitable admissible family $\mathcal{F}_\nu$ of length $< l$. If $\nu \neq \pi^{\infty}_{m+1,j}$, then by the inductive hypothesis on m we can find an $\mathcal{A}$-cell decomposition of $\nu(I_{r'}) \cap (S \cup S')$ compatible with the set $\{g = 0\}$, where $g = f \restriction S \cup S'$. To obtain the required $\mathcal{A}$-cell decomposition of $\nu(I_{r'}) \setminus (S \cup S')$ we take the direct image under ν of all the $\mathcal{A}$-cells of an $\mathcal{A}$-cell decomposition of $I_r \setminus (S \cup S')$ compatible with the set $\{f \circ \nu = 0\}$ (which can be obtained by the inductive hypothesis on l). Finally, if $\nu = \pi^{\infty}_{m+1,j}$, then the required $\mathcal{A}$-cell decomposition of $\nu(I_{r'})$ is given just by the sign of y.

2 Proof of theorem B

We recall the definition and main properties of generalised power series (see [DS98] for more details).

Let $m \in \mathbb{N}$. A set $S \subset [0, \infty)^m$ is called *good* if S is contained in a cartesian product $S_1 \times \ldots \times S_m$ of well ordered subsets of $[0, \infty)$. If S is a good set, define $S_{\min}$ as the set of minimal elements of S. By [DS98, Lemma 4.2], $S_{\min}$ is finite.

A *formal generalised power series* has the form

$$F(X) = \sum_{\alpha} c_\alpha X^\alpha,$$

where $\alpha = (\alpha_1, \ldots, \alpha_m) \in [0, \infty)^m$, $c_\alpha \in \mathbb{R}$ and X^α denotes the formal monomial $X_1^{\alpha_1} \cdot \ldots \cdot X_m^{\alpha_m}$, and the *support of* F $\operatorname{Supp}(F) := \{\alpha : \ c_\alpha \neq 0\}$ is a good set. These series are added the usual way and form an $\mathbb{R}$-algebra denoted by $\mathbb{R}[\![X^*]\!]$.

Let $\mathcal{G} \subseteq \mathbb{R}[\![X^*]\!]$ be a family of series such that the *total support* $\operatorname{Supp}(\mathcal{G}) := \bigcup_{F \in \mathcal{G}} \operatorname{Supp}(F)$ is a good set. Then $\operatorname{Supp}(\mathcal{G})_{\min}$ is finite and we denote by $\mathcal{G}_{\min} := \{X^\alpha : \ \alpha \in \operatorname{Supp}(\mathcal{G})_{\min}\}$ the *set of minimal monomials* of $\mathcal{G}$.

Let $m, n \in \mathbb{N}$ and $(X,Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$. We define $\mathbb{R}[\![X^*, Y]\!]$ as the subring of $\mathbb{R}[\![(X,Y)^*]\!]$ consisting of those series F such that $\mathrm{Supp}(F) \subset [0,\infty)^m \times \mathbb{N}^n$. Since $\mathbb{R}[\![X^*, Y]\!] \subseteq \mathbb{R}[\![X^*]\!][\![Y]\!]$, we say that the variables X are *generalised* and that the variables Y are *analytic.*

Let $r = (s,t) \in (0,\infty)^m \times (0,\infty)^n$ be a polyradius. A series $\sum c_{\alpha\beta} X^\alpha Y^\beta$ belongs to $\mathbb{R}\{X^*, Y\}_r$ if $\sum |c_{\alpha\beta}| s^\alpha t^\beta < \infty$. We denote by $\mathbb{R}\{X^*, Y\} := \bigcup_r \mathbb{R}\{X^*, Y\}_r$ the subalgebra of $\mathbb{R}[\![X^*, Y]\!]$ of all *convergent* generalised power series. An element $\sum c_{\alpha\beta} X^\alpha Y^\beta$ of $\mathbb{R}\{X^*, Y\}_r$ induces a function $f(x,y) := \sum c_{\alpha\beta} x^\alpha y^\beta$ defined on the set

$$I_{m,n,r} := [0, s_1) \times \ldots \times [0, s_m) \times (-t_1, t_1) \times \ldots \times (-t_n, t_n),$$

which is continuous on $I_{m,n,r}$ and analytic on $\mathring{I}_{m,n,r}$. We denote by $\mathbb{R}\{x^*, y\}_r$ the collection of such functions and define $\mathbb{R}\{x^*, y\} := \bigcup_r \mathbb{R}\{x^*, y\}_r$.

As in the previous section, the first step towards the proof of Theorem B is to give a monomialisation algorithm for the elements of $\mathbb{R}\{X^*, Y\}$ which respects a given variable. Our elementary transformations are still ramifications, translations, linear transformations and blow-up charts. However, due to the different nature of the variables X and Y, we need to be more specific about the action of such elementary transformations and how we are allowed to compose them. For example, $\mathbb{R}\{X^*, Y\}$ is not closed under arbitrary compositions, so we will only allow translations and linear transformations which involve the analytic variables. Moreover, we need to allow real exponents in the ramifications of the generalised variables. Finally, the action of some blow-up charts change the nature (from generalised to analytic or vice versa) of the variables involved. A further trivial transformation, a reflection, is required in order to be able to consider any given variable as generalised and hence to be able to ramify it with a real exponent.

Definition 2.1. Let $m, n, m', n' \in \mathbb{N}$ with $m+n = m'+n'$ and let $(X,Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$, $(X', Y') = (X'_1, \ldots, X'_{m'}, Y'_1, \ldots, Y'_{n'})$. An *elementary transformation* is a map $(X', Y') \mapsto (X, Y)$ of either of the following forms.

- A *ramification*: let $m = m', n = n'$, $\gamma \in \mathbb{R}^{>0}$ and $1 \le i \le m$, and set
$$r_i^\gamma(X', Y') = (X, Y), \quad \text{where } \begin{cases} X_k = X'_k & 1 \le k \le m,\ k \ne i \\ X_i = X_i'^\gamma & \\ Y_k = Y_k & 1 \le k \le n \end{cases}.$$

- A *Tschirnhausen translation*: let $m = m', n = n'$ and $H \in \mathbb{R}\left\{X'^*, Y'_1, \ldots, Y'_{n-1}\right\}$, with $H(0) = 0$, and set
$$\tau_H(X', Y') = (X, Y), \quad \text{where } \begin{cases} X_k = X'_k & 1 \le k \le m \\ Y_n = Y'_n + H\left(X', Y'_1, \ldots, Y'_{n-1}\right) & \\ Y_k = Y'_k & 1 \le k \le n-1 \end{cases}.$$

- A *linear transformation*: let $m = m', n = n'$, $1 \le i \le n$ and $c = (c_1, \ldots, c_{i-1}) \in \mathbb{R}^{i-1}$, and set
$$L_{i,c}(X', Y') = (X, Y), \quad \text{where } \begin{cases} X_k = X'_k & 1 \le k \le m \\ Y_k = Y'_k & i \le k \le n \\ Y_k = Y'_k + c_k Y'_i & 1 \le k < i \end{cases}.$$

- A *blow-up chart,* i.e. either of the following maps:
  - for $1 \le j < i \le m$ and $\lambda \in (0,\infty)$, let $m' = m-1$ and $n' = n+1$ and set
$$\pi_{i,j}^\lambda(X', Y') = (X, Y), \quad \text{where } \begin{cases} X_k = X'_k & 1 \le k < i \\ X_i = X'_j(\lambda + Y'_1) & \\ X_k = X'_{k-1} & i < k \le m \\ Y_k = Y'_{k+1} & 1 \le k \le n \end{cases};$$

– for $1 \leq j, i \leq m$, let $m' = m$ and $n' = n$, and set

$$\pi^0_{i,j}(X', Y') = (X, Y), \quad \text{where} \begin{cases} X_k = X'_k & 1 \leq k \leq m,\ k \neq i \\ X_i = X'_j X'_i & \\ Y_k = Y'_k & 1 \leq k \leq n \end{cases}$$

and $\pi^\infty_{i,j} = \pi^0_{j,i}$;

– for $1 \leq i \leq n,\ 1 \leq j \leq m$ and $\lambda \in \mathbb{R}$, let $m' = m$ and $n' = n$, and set

$$\pi^\lambda_{m+i,j}(X', Y') = (X, Y), \quad \text{where} \begin{cases} X_k = X'_k & 1 \leq k \leq m \\ Y_i = X'_j (\lambda + Y'_i) & \\ Y_k = Y'_k & 1 \leq k \leq n,\ k \neq i \end{cases};$$

– for $1 \leq i \leq n,\ 1 \leq j \leq m$, let $m' = m+1$ and $n' = n-1$, and set

$$\pi^{\pm\infty}_{m+i,j}(X', Y') = (X, Y), \quad \text{where} \begin{cases} X_k = X'_k & 1 \leq k \leq m,\ k \neq j \\ X_j = X'_{m+1} X'_j & \\ Y_k = Y'_k & 1 \leq k < i \\ Y_i = \pm X'_{m+1} & \\ Y_k = Y'_{k-1} & i < k \leq n \end{cases}.$$

- A *reflection*: let $m' = m+1$, $n' = n-1$ and $1 \leq i \leq n$, and set

$$\sigma^\pm_{m+i}(X', Y') = (X, Y), \quad \text{where} \begin{cases} X_k = X'_k & 1 \leq k \leq m \\ Y_k = Y'_k & 1 \leq k < i \\ Y_i = \pm X'_{m+1} & \\ Y_k = Y'_{k-1} & i < k \leq n \end{cases}.$$

Let $k \leq 1$ and for all $i \in \{1, \ldots, k\}$ let $\nu_i : \left(X'_{(i)}, Y'_{(i)}\right) \mapsto \left(X_{(i)}, Y_{(i)}\right)$ be an elementary transformation, where $X'_{(i)}$ is an m'_i-tuple, $Y'_{(i)}$ is an n'_i-tuple, $X_{(i)}$ is an m_i-tuple and $Y_{(i)}$ is an n_i-tuple, with $m'_i + n'_i = m_i + n_i$. If $k = 1$ or if $k > 1$ and $m_i = m'_{i-1}$ for all $i = 1, \ldots, k$, then we say that $\rho := \nu_1 \circ \ldots \circ \nu_k$ is an *admissible transformation*. One can verify (see [DS98]) that if $\rho : (X', Y') \mapsto (X, Y)$ is an admissible transformation and $F \in \mathbb{R}\{X^*, Y\}$, then $F \circ \rho \in \mathbb{R}\{X'^*, Y'\}$. Moreover, if $\mathcal{G} \subseteq \mathbb{R}\{X^*, Y\}$ is a collection with good total support, then the collection $\{F \circ \rho :\ F \in \mathcal{G}\}$ has good total support.

An *elementary family* is either of the following collections of elementary transformations: $\{r^\gamma_i\}$ (for some $1 \leq i \leq m$), $\{\sigma^+_{m+i}, \sigma^-_{m+i}\}$ (for some $1 \leq i \leq n$), $\{\tau_H\}$, $\{L_{i,c}\}$ (for some $1 \leq i \leq n$), $\{\pi^\lambda_{i,j} :\ \lambda \in [0, \infty]\}$ (for some $1 \leq i, j \leq m$), or $\{\pi^\lambda_{m+i,j} :\ \lambda \in \mathbb{R} \cup \{\pm\infty\}\}$ (for some $1 \leq i \leq n,\ 1 \leq j \leq m$). An *admissible family* is defined inductively. An admissible family of length 1 is an elementary family. An admissible family $\mathcal{F}$ of length $\leq q$ is obtained from an elementary family $\mathcal{F}_0$ in the following way: for all $\nu \in \mathcal{F}_0$, let $\mathcal{F}_\nu$ be an admissible family of length $\leq q-1$ such that $\forall \rho' \in \mathcal{F}_\nu,\ \nu \circ \rho'$ is an admissible transformation and define $\mathcal{F} = \{\nu \circ \rho' :\ \nu \in \mathcal{F}_0,\ \rho' \in \mathcal{F}_\nu\}$.

Finally, we say that a series $F \in \mathbb{R}\{X^*, Y\}$ has a certain property P *after admissible family* if there exists ad admissible family $\mathcal{F}$ such that for every $\rho \in \mathcal{F}$ the series $F \circ \rho(X', Y')$ has the property P.

2.2. A series $F \in \mathbb{R}\{X^*, Y\}$ is *normal* if there are $\alpha \in [0, \infty)^m,\ \beta \in \mathbb{N}^n$ and a unit $U \in \mathbb{R}\{X^*, Y\}^\times$ such that $F(X) = X^\alpha Y^\beta U(X, Y)$.

Let $(X, Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ and $F \in \mathbb{R}\{X^*, Z^*, Y\}$. We show that, after admissible family, F is normal.

Write $F(X, Y, Z) = \sum_{\alpha \in A} F_\alpha(X, Y) Z^\alpha$ and consider the family $\mathcal{G} = \{F_\alpha(X, Y) :\ \alpha \in A\} \subseteq \mathbb{R}\{X^*, Y\}$. Note that $A \subseteq [0, \infty)$ is a well ordered set and $\mathcal{G}$ has good total support.

Step 1: simplifying the Newton polygon of $\mathcal{G}$ in the variables X. We show that, after admissible family, there are $\beta \in [0, \infty)^m$ and a collection $\{G_\alpha(X,Y) : \alpha \in A\} \subseteq \mathbb{R}\{X^*, Y\}$ such that $\forall \alpha \in A,\ F_\alpha(X,Y) = X^\beta G_\alpha(X,Y)$ and $G_{\alpha_0}(0,Y) \not\equiv 0$, for some $\alpha_0 \in A$.

In order to do this, we view $\mathcal{G}$ as a subset of $\mathbb{A}[\![X^*]\!]$, with $\mathbb{A} = \mathbb{R}[\![Y]\!]$. In [DS98, 4.11] the authors define the *blow-up height* of a finite set of monomials, denoted by b_X. It follows from the definition of b_X that if $b_X(\mathcal{G}_{\min}) = (0,0)$, then there exists $\beta \in [0,\infty)^m$ such that $\mathcal{G}_{\min} = \{X^\beta\}$, which is what we want. The proof is by induction on the pairs $(m, b_X(\mathcal{G}_{\min}))$, ordered lexicographically. If $m = 0$, there is nothing to prove. If $m = 1$, then $b_X(\mathcal{G}_{\min}) = (0,0)$.

Hence we may assume that $m > 1$ and $b_X(\mathcal{G}_{\min}) \neq (0,0)$. It follows from the proof of [DS98, Proposition 4.14] that there are $i, j \in \{1, \ldots, m\}$ and suitable ramifications $r_i^\gamma,\ r_j^\delta$ of the variables X_i and X_j such that, after the admissible transformations $\rho_0 := r_i^\gamma \circ r_j^\delta \circ \pi_{i,j}^0$ and $\rho_\infty := r_i^\gamma \circ r_j^\delta \circ \pi_{i,j}^\infty$, the blow-up height b_X of $\mathcal{G}_{\min}$ has decreased. Moreover, for every $\lambda \in (0,\infty)$, after the admissible transformation $\rho_\lambda := r_i^\gamma \circ r_j^\delta \circ \pi_{i,j}^\lambda$, the series in the family $\mathcal{G}$ have one less generalised variable and one more analytic variable, so m has decreased. Since admissible transformations preserve having good total support, the inductive hypothesis applies and we obtain the required conclusion.

The ring $\mathbb{R}[\![X^*, Y]\!]$ is clearly not Noetherian. However, the next step provides a finiteness property which is enough for our purposes. The proof takes inspiration from [Hor73, Theorem 6.3.3].

Step 2: quasi-Noetherianity. We show that, after admissible family, the $\mathbb{R}\{X^*, Y\}$-ideal generated by the collection $\mathcal{G}$ is finitely generated.

Notice first that, if this is the case, then for every $d \in \mathbb{N}$, after admissible family, the $\mathbb{R}\{X^*, Y\}$-module generated by the set $\{(F_{\alpha_1}, \ldots, F_{\alpha_d}) : \alpha_1, \ldots, \alpha_d \in A\}$ is also finitely generated (the proof of this statement is by induction on d as in [Hor73, Lemma 6.3.2]).

The proof of quasi-Noetherianity is by induction on the total number of variables $m + n$. If $m + n = 1$ then, since $\mathcal{G}$ has good total support, the ideal generated by $\mathcal{G}$ is principal. Hence suppose that $m + n > 1$. Recall that, by the previous step, there are $\beta \in [0,\infty)^m$ and a collection $\{G_\alpha(X,Y) : \alpha \in A\} \subseteq \mathbb{R}\{X^*, Y\}$ such that $\forall \alpha \in A,\ F_\alpha(X,Y) = X^\beta G_\alpha(X,Y)$ and $G_{\alpha_0}(0,Y) \not\equiv 0$, for some $\alpha_0 \in A$. After a linear transformation $L_{n,c}$, we may suppose that G_{α_0} is regular of some order d in the variable Y_n.

Let $\hat{Y} = (Y_1, \ldots, Y_{n-1})$. By Weierstrass Division (see [DS98, 5.10 (1)]), for every $\alpha \in A$ there are $Q_\alpha \in \mathbb{R}\{X^*, Y\}$ and $B_{\alpha,0}, \ldots, B_{\alpha,d-1} \in \mathbb{R}\left\{X^*, \hat{Y}\right\}$ such that $G_\alpha = G_{\alpha_0} Q_\alpha + R_\alpha$, where $R_\alpha(X,Y) = \sum_{i=0}^{d-1} B_{\alpha,i}\left(X, \hat{Y}\right) Y_n^i$. A careful analysis of the proof of [DS98, 4.17] shows that the total support of the collection $\mathcal{B} = \{B_\alpha = (B_{\alpha,0}, \ldots, B_{\alpha,d-1}) : \alpha \in A\}$ is contained in the good set $\Sigma\mathrm{Supp}(\mathcal{G})$ of all finite sums (done component-wise) of elements of $\mathrm{Supp}(\mathcal{G})$. Hence, by the inductive hypothesis and the remark at the beginning of this proof, after admissible family, the $\mathbb{R}\left\{X^*, \hat{Y}\right\}$-module generated by $\mathcal{B}$ is finitely generated. Therefore, there are $\alpha_1, \ldots, \alpha_q \in A$ and for all $\alpha \in A$ there are $C_{\alpha,1}, \ldots, C_{\alpha,q} \in \mathbb{R}\left\{X^*, \hat{Y}\right\}$ such that $B_\alpha = \sum_{j=1}^q C_{\alpha,j} B_{\alpha_j}$. Putting everything together, we obtain that, for every $\alpha \in A$,

$$F_\alpha = \left(Q_\alpha - \sum_{j=1}^q C_{\alpha,j} Q_{\alpha_j}\right) F_{\alpha_0} + \sum_{j=1}^q C_{\alpha,j} F_{\alpha_j}.$$

Step 3: finite presentation. We show that, after admissible family, there are $d \in \mathbb{N},\ \alpha_1 > \ldots > \alpha_d \in A$ and units $U_1, \ldots, U_d \in \mathbb{R}\{X^*, Z^*, Y\}^\times$ such that

$$F(X,Y,Z) = \sum_{i=1}^d F_{\alpha_i}(X,Y) Z^{\alpha_i} U_i(X,Y,Z). \tag{**}$$

To obtain this, since the ideal generated by $\mathcal{G}$ is finitely generated, we can apply the monomialisation algorithm for generalised power series in [MVSSR12] or in [RS13, Theorem 3.11] simultaneously to the generators and obtain that, after admissible family, the generators are linearly ordered by

division. Hence, there is $\alpha_1 \in A$ and for all $\alpha \in A$ there is $Q_\alpha \in \mathbb{R}\{X^*, Y\}$ such that $F_\alpha = F_{\alpha_1} \cdot Q_\alpha$. This allows us to write

$$F(X,Y,Z) = \sum_{\alpha<\alpha_1} F_\alpha(X,Y) Z^\alpha + F_{\alpha_1}(X,Y) Z^{\alpha_1} U(X,Y,Z),$$

where $U(X,Y,Z) = 1 + \sum_{\alpha>\alpha_1} Q_\alpha(X,Y) Z^{\alpha-\alpha_1}$ is a unit of $\mathbb{R}\{X^*, Z^*, Y\}$. Notice that all the admissible families involved in the proof so far act as the identity on the variable Z. Hence we can consider the series $G(X,Y,Z) = \sum_{\alpha<\alpha_1} F_\alpha(X,Y) Z^\alpha \in \mathbb{R}\{X^*, Z^*, Y\}$ and repeat the above steps for G. This procedure will provide, after admissible family, a decreasing sequence $\alpha_1 > \alpha_2 > \ldots$ which is necessarily finite, since A is well-ordered.

Step 4: decreasing the number of terms. We show that, after admissible family, either the variable Z is analytic (i.e. $F \in \mathbb{R}\{X^*, Y, Z\}$) or the number of terms in the finite presentation (**) has decreased. More precisely, we say that F has a finite presentation of order d if there are $H_1, \ldots, H_d \in \mathbb{R}\{X^*, Y\}$ and units $U_1, \ldots, U_d \in \mathbb{R}\{X^*, Z^*, Y\}^\times$ such that $F(X,Y,Z) = H_1(X,Y) H(X,Y,Z)$, where

$$H(X,Y,Z) = Z^{\alpha_1} U_1(X,Y,Z) + H_2(X,Y) Z^{\alpha_2} U_2(X,Y,Z) + \ldots + H_d(X,Y) Z^{\alpha_d} U_d(X,Y,Z).$$

Notice that, since $\forall \alpha \in A\ F_{\alpha_1} | F_\alpha$, we know that F has a finite presentation of order d. Our aim is to show that, after admissible family, either Z is analytic or F has a finite presentation of order $< d$.

In what follows, up to suitable reflections, we will always consider the variables (X,Y) as generalised, hence we may suppose $Y = \emptyset$. We can apply the monomialisation algorithm in [MVSSR12] simultaneously to $H_2, \ldots, H_d$ in such a way that, after admissible family, we have

$$H(X,Z) = Z^{\alpha_1} \tilde{U}_1(X,Z) + X^{\Gamma_2} Z^{\alpha_2} \tilde{U}_2(X,Z) + \ldots + X^{\Gamma_d} Z^{\alpha_d} \tilde{U}_d(X,Z),$$

for some units $\tilde{U}_i \in \mathbb{R}\{X^*, Z^*\}$, and the exponents $\Gamma_i = \left(\gamma_i^{(1)}, \ldots, \gamma_i^{(m)}\right)$ are such that the monomials $X^{\frac{\Gamma_i}{\alpha_1-\alpha_i}}$ are linearly ordered by division. Let $i_0 \in \{2, \ldots, d\}$ be smallest with the property that

$$\forall i \in \{2, \ldots, d\},\ \forall j \in \{1, \ldots, m\},\ \frac{\gamma_{i_0}^{(j)}}{\alpha_1 - \alpha_{i_0}} \leq \frac{\gamma_i^{(j)}}{\alpha_1 - \alpha_i}. \qquad (\#)$$

Suppose $\gamma_{i_0}^{(1)} \neq 0$ and perform a ramification of the variable X_1 with exponent $\gamma := \frac{\gamma_{i_0}^{(1)}}{\alpha_1-\alpha_{i_0}}$. We consider the family of blow-up transformations $\left\{\pi_{m+1,1}^\lambda : \lambda \in [0,\infty]\right\}$. If $\lambda \in (0,\infty)$, then after the transformation $\pi_{m+1,1}^\lambda$, the variable Z is analytic. After the transformation $\pi_{m+1,1}^\infty$, we can write

$$H(X,Z) = Z^{\alpha_1} \left[\tilde{U}_1(X,Z) + X^{\Gamma_2} Z^{\beta_2} \tilde{U}_2(X,Z) + \ldots + X^{\Gamma_d} Z^{\beta_d} \tilde{U}_d(X,Z)\right],$$

where $\beta_i := \frac{\gamma_i^{(1)}}{\gamma_{i_0}^{(1)}}(\alpha_1 - \alpha_{i_0}) + \alpha_i - \alpha_1$ is nonnegative, thanks to $(\#)$. Notice that, since by $(\#)$ every $\gamma_i^{(1)}$ is nonzero, the expression between square brackets is a unit. Hence in this case F has a finite presentation of order 1. After the transformation $\pi_{m+1,1}^0$, we can write

$$H(X,Z) = X_1^{\gamma\alpha_1} \left[Z^{\alpha_1} \tilde{U}_1(X,Z) + X^{\Delta_2} Z^{\alpha_2} \tilde{U}_2(X,Z) + \ldots + X^{\Delta_d} Z^{\alpha_d} \tilde{U}_d(X,Z)\right],$$

where $\Delta_i = \left(\delta_i^{(1)}, \ldots, \delta_i^{(m)}\right) := \left(\gamma_i^{(1)} - \gamma_{i_0}^{(1)} \frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_{i_0}}, \gamma_i^{(2)}, \ldots, \gamma_i^{(m)}\right)$. Notice that, by $(\#)$, the exponents $\delta_i^{(1)}$ are nonnegative and $\delta_{i_0}^{(1)} = 0$. Hence, up to factoring out by a power of X_1, the variable X_1 does not appear any more in the i_0^{th} term of the above finite presentation. By repeating this step with the other variables X_j such that $\gamma_{i_0}^{(j)} \neq 0$, we obtain

$$H(X,Z) = X^\Delta \left[Z^{\alpha_{i_0}} V(X,Z) + X^{\Delta'_{i_0+1}} Z^{\alpha_{i_0+1}} \tilde{U}_{i_0+1}(X,Z) + \ldots + X^{\Delta'_d} Z^{\alpha_d} \tilde{U}_d(X,Z)\right],$$

where $V \in \mathbb{R}\{X^*, Z^*\}^\times$, the components of Δ are $\frac{\alpha_1\gamma_{i_0}^{(j)}}{\alpha_1-\alpha_{i_0}}$ and the components of Δ'_i are $\gamma_i^{(j)} - \gamma_{i_0}^{(j)} \frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_{i_0}}$. Hence F has a finite presentation of order $d - i_0 + 1$.

We have reduced to the case when either the variable Z is analytic, or F has a finite presentation of order 1. In this latter case, we can apply the monomialisation algorithm in [MVSSR12] to the series $H_1(X, Y)$ and finish the proof of 2.2. If Z is analytic, then we perform the first three steps in this proof and obtain a finite presentation for F. Note that, since Z is analytic, this implies that F is regular in the variable Z. We conclude the proof of 2.2 in this case by showing how to decrease the order of regularity. We argue essentially as in Step 2 of the proof of 1.2, up to suitable reflections and ramifications (the details can be found in [RS13, Proof of Theorem 3.11]). This concludes the proof of 2.2.

To finish the proof of Theorem B we argue as in 1.4. A formal admissible transformation $\rho : (X', Y') \mapsto (X, Y)$ induces a geometric transformation $\rho : I_{m',n',r'} \to I_{m,n,r}$ (for some polyradii r', r) and the formal monomialisation algorithm for the series $F(X, Y, Z) \in \mathbb{R}\{X^*, Z^*, Y\}_r$ implies a geometric monomialisation result for the function $F(x, y, z) \in \mathbb{R}\{x^*, z^*, y\}_r$. Since the monomialisation process respects the variable z, we can conclude as in 1.4 that, if $\mathcal{A}$ is as in the statement of Theorem B, then there exist a neighbourhood $W \subseteq \mathbb{R}^{m+n+1}$ of the origin and an $\mathcal{A}$-cell decomposition of $W \cap I_{m,n,r}$ which is compatible with the set $\{(x, y, z) \in W \cap I_{m,n,r} : F(x, y, z) = 0\}$.

3 Proof of Theorems C and D

Theorem C can be proven by a combination of the arguments in the proofs of Theorem A and Theorem B.

For $m, n \in \mathbb{N}$ and $r \in (0, \infty)^{m+n}$, we consider the $\mathbb{R}$-algebra of functions $\mathcal{G}_{m,n,r}$ defined in [DS00, Definition 6.1]. The elements of this algebra are functions defined on $I_{m,n,r}$, in the notation of Section 2. We denote by $x = (x_1, \ldots, x_m)$ the variables which are only allowed to take positive values, also called *Gevrey variables*, and by $y = (y_1, \ldots, y_n)$ the *convergent variables.* The functions in $\mathcal{G}_{m,n,r}$ are $\mathcal{C}^\infty$ and the map which associates to the germ at zero of $f(x, y) \in \mathcal{G}_{m,n,r}$ its Taylor expansion at zero $\hat{f}(X, Y) \in \mathbb{R}[\![X, Y]\!]$, is injective. We denote by $\mathbb{R}\{X; Y\}_{\mathcal{G},r} \subseteq \mathbb{R}[\![X, Y]\!]$ the image of the Taylor map and let $\mathbb{R}\{X; Y\}_{\mathcal{G}} = \bigcup_{r \in (0,\infty)^{m+n}} \mathbb{R}\{X; Y\}_{\mathcal{G},r}$. The collection $\left\{\mathbb{R}\{X; Y\}_{\mathcal{G}} : m, n \in \mathbb{N}\right\}$, despite the fact that its elements are power series with natural exponents, behaves in a similar way to the collection of all convergent generalised power series, in that it is closed under the elementary transformations in Definition 2.1 (if we allow only ramifications with natural exponents) and monomial division (see [DS00]). Hence a simplified version of the proof of 2.2, which takes into account the fact that the exponents of the series are natural numbers, goes through in this setting, by replacing every instance of $\mathbb{R}\{X^*, Y\}$ with $\mathbb{R}\{X; Y\}_{\mathcal{G}}$.

Theorem D can be proven by the same arguments in the proof of Theorem B.

For $m, n \in \mathbb{N}$ and $r \in (0, \infty)^{m+n}$, we consider the $\mathbb{R}$-algebra of functions $\mathcal{Q}_{m,n,r}$ defined in [KRS09, Definition 7.1]. The elements of this algebra are functions defined on $I_{m,n,r}$, in the notation of Section 2. The functions in $\mathcal{Q}_{m,n,r}$ are continuous and the map T (see [KRS09, Definition 2.6]) which associates to the germ at zero of $f(x, y) \in \mathcal{Q}_{m,n,r}$ its asymptotic expansion at zero $T(f)(X, Y) \in \mathbb{R}[\![X^*, Y]\!]$, is injective. We denote by $\mathbb{R}\{X^*, Y\}_{\mathcal{Q},r} \subseteq \mathbb{R}[\![X^*, Y]\!]$ the image of the morphism T and let $\mathbb{R}\{X^*, Y\}_{\mathcal{Q}} = \bigcup_{r \in (0,\infty)^{m+n}} \mathbb{R}\{X^*, Y\}_{\mathcal{Q},r}$. The proof of 2.2 goes through in this setting, by replacing every instance of $\mathbb{R}\{X^*, Y\}$ with $\mathbb{R}\{X^*, Y\}_{\mathcal{Q}}$. In fact, the collection $\left\{\mathbb{R}\{X^*, Y\}_{\mathcal{Q}} : m, n \in \mathbb{N}\right\}$ is closed under the elementary transformations in Definition 2.1 and monomial division (see [KRS09]). Moreover, it is closed under truncation (see [KRS09, Definition 1.2 and Proposition 5.6]), hence in Step 1 the coefficients $F_\alpha(X, Y)$ of $F(X, Y, Z)$ are in $\mathbb{R}\{X^*, Y\}_{\mathcal{Q}}$ and the series $G(X, Y, Z)$ in Step 3 is in $\mathbb{R}\{X^*, Z^*, Y\}_{\mathcal{Q}}$.

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